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BRANCHED EXTENSIONS OF CURVES IN COMPACT SURFACES

BY

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ABSTRACT. A *polymersion* is a map F: $M \to N$ where M and N are compact surfaces, orientable or nonorientable, M a surface with boundary, where

(a) At each interior point of M, there is an integer $n \geqslant 1$ such that F is topologically equivalent to the complex map z^n in a neighborhood about the point.

(b) At each point x in the boundary of M, δM, there is a neighborhood U containing x such that U is homeomorphic to $F(U)$.

A normal polymersion is one where $F(\delta M)$ is a normal set of curves in N. We are concerned with establishing a combinatorial representation for normal polymersions which map to arbitrary compact surfaces.

1. Introduction. A *polymersion* is a map F: $M \to N$ where M and N are smooth, compact surfaces, orientable or nonorientable, M a surface with boundary where

(a) At each interior point of M, there is an integer $n \geqslant 1$ such that F is topologically equivalent to the complex map z^n in a neighborhood about the point. A point where $n > 1$ is a *critical point of multiplicity* $n - 1$ and its image a *branch point*.

(b) At each point x in the boundary of M, δM, there is a neighborhood U containing x such that U is homeomorphic to $F(U)$.

We are concerned with polymersions (*normal polymersions*) where $F(\delta M)$ is a *normal set* of regular curves; i.e., a set of regular curves possessing a finite number of transverse self-intersections. If $f = \{f_1, f_2, \ldots, f_\rho\}$ is a normal set of curves in N and if there is a polymersion F: $M \to N$ with $F(\delta M_i) = f_i$, $i = 1, \ldots, \rho$ (where δM_i, $i = 1, \ldots, \rho$, are the components in δM), we call F an *extension of* f.

This paper extends the results of [3] to include polymersions between nonorientable surfaces. The reader needs a familiarity with [3]. For background information and a history of the kinds of problems considered here, also see [3].

2. Assemblages. Let $f = \{f_1, \ldots, f_\rho\}$ be a normal set of closed, two-sided (i.e. orientation preserving), directed (i.e. oriented) curves in a closed surface N. Arbitrarily select one side of each curve to be the *inside*. In the diagrams to follow, the marked side of the curves will be the inside. If N is orientable and oriented, then selecting the inside to lie to the left of the curves will make the results here coincide with earlier works done for orientable surfaces.

The set f can be decomposed into a collection of simple closed curves called *Gaussian circles* by separating and smoothing at the intersection points so that the

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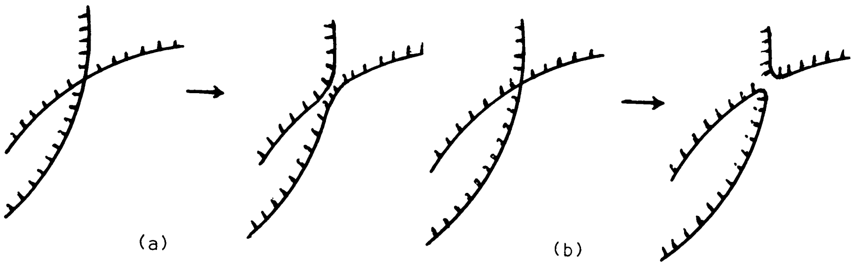
choice of inside on f determines an inside on the Gaussian circles as illustrated in Figure 1 below. The separation is made as in (a); not as in (b).



(a) (b)

FIGURE 1

Select a point ∞ in $N - f$. A Gaussian circle which bounds a disk not containing ∞ is *positive* (*negative*) if the disk lies to the inside (outside) of the curve. A curve bounded by a disk is *positively turning* if it has no negative Gaussian circles.

We associate with the curve set f a set of data $A_f = (R, W, S, P)$ called an *assemblage on f*. The assemblage is essentially the same as in **[2]** and **[3]** but extended to make sense for curves in arbitrary closed surfaces.

Choose a minimal generating set for the fundamental group of N based at ∞, $\Pi(N, \infty) = \Pi$. Choose a set of directed curves α which are representatives for the generators of Π so that $N - \alpha$ is a disk. A ray is either a member of α or a directed arc initiating in $N - \alpha$ and terminating at ∞. A ray in α is called a *fundamental ray*. A *raying for f*, R, is a set of rays which (i) includes α, (ii) do not intersect each other, and (iii) intersect f transversely. A raying is *sufficient* if every component curve and every negative Gaussian circle is crossed.

A *wheel* is a directed simple closed curve bounding a disk on its outside containing ∞ but no points on f. Choose a (possibly empty) set of wheels $W = \{w_1, \ldots, w_\beta\}$ such that the wheels are concentric.

The points in the set $C = R \cap (f \cup W)$ are called crossings. A crossing is *positive* (*negative*) if the ray crosses the curve from inside to outside (outside to inside). We use the letter ν for the number of negative crossings. A curve is said to satisfy the *generator parity condition*, GPC, if the number of negative crossings on any fundamental ray equals the number of positive crossings on that ray.

Define the *successor permutation* S on the crossing set C to be the permutation that maps a crossing on a component (an f_i or w_i) to the next crossing in the direction of the orientation. Define an *assembling permutation* P, also on C, such that (i) if x is a crossing on a ray r, then xP is also on r and (ii) if x is a negative crossing on r, then $xP = y$ is positive and $yP = x$. If for every negative crossing x in C, xP is mapped closer (relative to the orientation on the ray) than x to ∞, then P and the assemblage are said to be *effective*. The cycles of P are classified into *pairs* and *fans*, the pairs being the cycles which contain negative crossings. The number of positive crossings minus the number of fans is called the *partial branching number* π of the assemblage. We say an assemblage is *transitive* if S and P generate a transitive group.

Let x be a crossing and select an oriented neighborhood U of x. We say the *auxiliary sign* of x, $\lambda(x)$, is positive (negative) if the curve crosses at x from right to left (left to right) relative to the orientation on U (viewed to be counterclockwise).

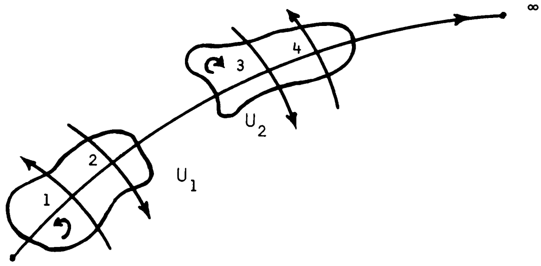


FIGURE 2

In Figure 2 crossings 1 and 3 have positive auxiliary signs and 2 and 4 have negative signs relative to the orientations on the neighborhoods U_1 and U_2.

Let $C^* = C \times \{-1, 1\}$. If $x \in C$, then x in C^* will refer to $(x, 1)$ and $\bar{x}$ to $(x, -1)$. Define $\bar{\bar{x}}$ to be x. Each crossing now has two names. For each pair and fan choose an oriented neighborhood containing all the crossings thus establishing the auxiliary signs. We use these signs to define permutations S^* and P^* on C^* as described below.

(i) If $xS = y$, let $xS^* = y$ and $\bar{y}S^* = \bar{x}$.

(ii) If (xy) is a pair in P, let

$(xy)(\bar{x}\bar{y})$ be cycles in P^* if $\lambda(x) \neq \lambda(y)$ and

$(x\bar{y})(\bar{x}y)$ be cycles in P^* if $\lambda(x) = \lambda(y)$.

(iii) If x and y are part of a fan with $xP = y$, then

$xP^* = y$ and $\bar{y}P^* = \bar{x}$ if $\lambda(x) = \lambda(y) = +1$,

$\bar{x}P^* = \bar{y}$ and $yP^* = x$ if $\lambda(x) = \lambda(y) = -1$,

$xP^* = \bar{y}$ and $yP^* = \bar{x}$ if $\lambda(x) = +1$ and $\lambda(y) = -1$, and

$\bar{x}P^* = y$ and $\bar{y}P^* = x$ if $\lambda(x) = -1$ and $\lambda(y) = +1$.

Note that in the case of the pairs, P^* depends only upon whether or not the auxiliary signs are matched and hence is independent of the choice of orientation on the neighborhood.

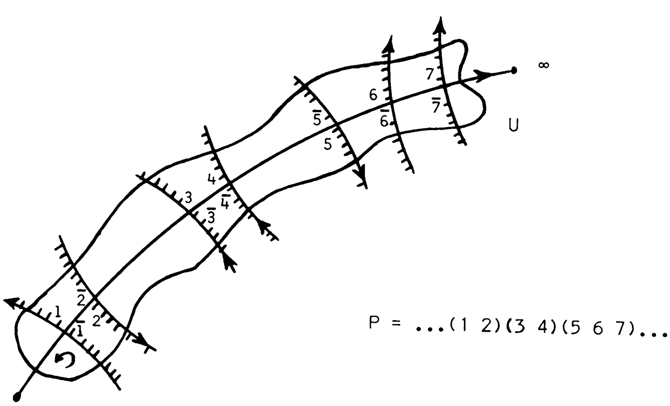


FIGURE 3

In Figure 3 the auxiliary signs taken relative to the orientation on U are

$$\lambda(1) = \lambda(3) = \lambda(4) = \lambda(6) = \lambda(7) = +1, \quad \text{and} \quad \lambda(2) = \lambda(5) = -1.$$

Hence the part of P^* determined by these signs and P is

$$P^* = \ldots (1\ 2)(\bar{1}\ \bar{2})(3\ \bar{4})(\bar{3}\ 4)(\bar{5}\ 6\ 7)(\bar{7}\ \bar{6}\ 5) \ldots .$$

An *orbit* of the assemblage is a closed curve in N obtained by using the permutations S^* and P^* as described below. Pick any crossing x (represented as x or $\bar{x}$ in C^*) and follow the curve from x to xS^*. If xS^* is part of a pair, proceed from xS^* on the ray containing xS^* to xS^*P^*. If xS^* is part of a fan, follow the ray to its base and back up the ray to xS^*P^*. Repeat the process by applying S^* and P^* to the new crossing xS^*P^*. Continue the path in this manner until it terminates at the starting point x. The same curve traversed in the opposite direction will be obtained by beginning at $\overline{xS^*}$. Thus each orbit is described by two cycles in the permutation S^*P^*. We use the letter ζ for the total number of orbits.

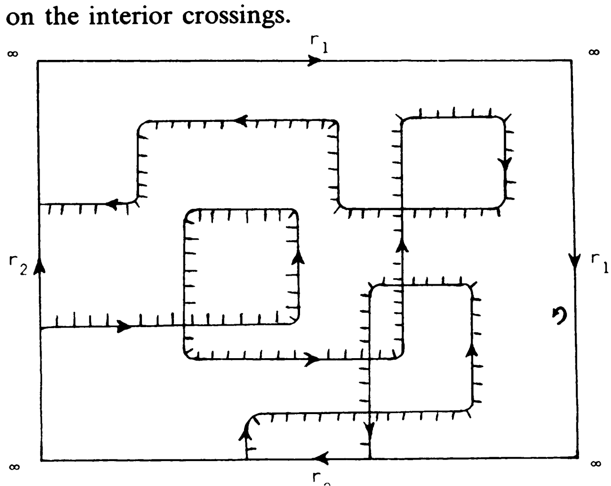
The orbits may be neither regular nor normal but can be approximated arbitrarily closely by regular normal curves. The inside of the approximating curves should be chosen to agree with the inside of f on the segments where they overlap. Define an orbit to be *positively turning* if there is a sequence of positively turning regular normal curves approaching the orbit as a limit. An orbit is *simple* if there is an approximating sequence of simple regular normal closed curves. An assemblage is *simple* if all its orbits are simple.

Note that if there are no fans on the fundamental rays, then the orbits cannot cross these rays. Thus we have the following observation.

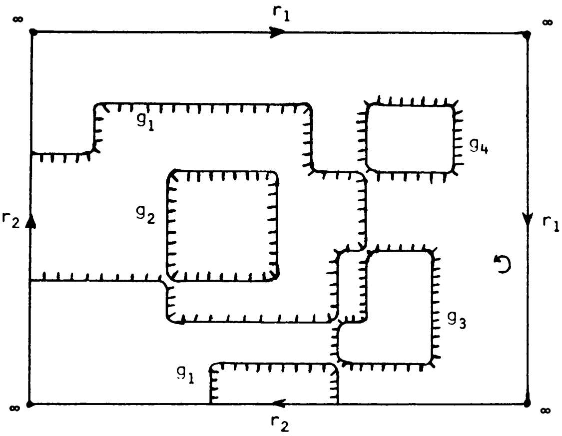
(1) If f is a curve satisfying GPC and A_f is an effective assemblage on a sufficient raying for f, then the orbits of A_f are positively turning and lie in the disk $N - \alpha$.

Note finally that if N is oriented and one directs $f \cup W$ so that the inside of the curves always lies to the left, then the action of S^* and P^* on C^* is the same as that of S and P on C (and its mirror image) in [2] and [3].

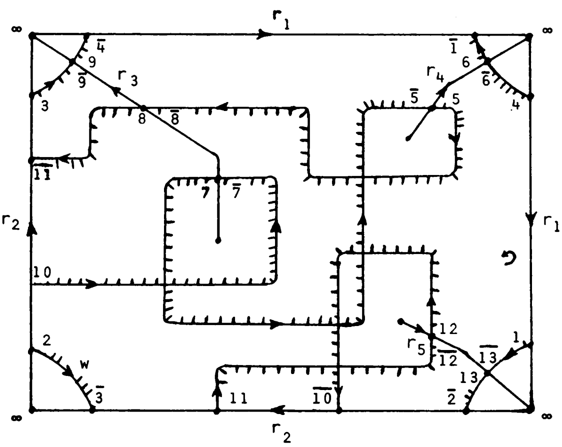
2.1. Example. Consider the curve f in the Klein bottle pictured below. We will use the orientation on the disk $N - \alpha$ indicated in the picture to determine the auxiliary signs on the interior crossings.



Separating at the intersection points yields four Gaussian circles, g_1, g_2, g_3, g_4. The first two, g_1 and g_2, are positive and g_3 and g_4 are negative.



Add rays r_3, r_4, and r_5 to produce a sufficient raying $R = \{r_1, r_2, r_3, r_4, r_5\}$ and add one wheel w as pictured below.



The crossings are numbered 1 through 13. Crossings 2, 4, 5, 10, and 12 are negative and the rest are positive. Labelling the crossings as above (with the unbarred letter on the inside corner following the crossing in the direction of the curve) helps in reading the orbits.

The successor permutation S and a choice for the assembling permutation P are below.

$$S = (1\ 13\ 2\ 3\ 9\ 4\ 6)(10\ 7\ 5\ 8\ 11\ 12),$$
$$P = (4\ 1)(5\ 6)(12\ 13)(7\ 8)(9)(10\ 11)(2\ 3).$$

The cycles (7 8) and (9) in P are fans and the rest are pairs. Note that P is effective and S and P generate a transitive group.

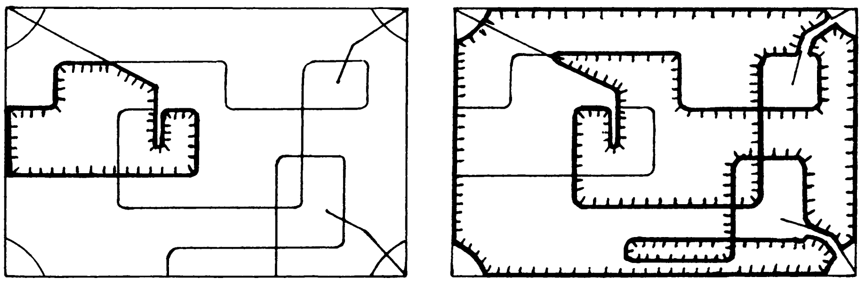
Pairs (2 3) and (1 4) are the only ones where the auxiliary signs are matched. The auxiliary signs on the crossings in the fan (7 8) are both positive. Thus S^*, P^*, and S^*P^* are determined to be the following.

$$S^* = (1\ 13\ 2\ 3\ 9\ 4\ 6)(\bar{6}\ \bar{4}\ \bar{9}\ \bar{3}\ \bar{2}\ \overline{13}\ \bar{1})(10\ 7\ 5\ 8\ 11\ 12)(\overline{12}\ \overline{11}\ \bar{8}\ \bar{5}\ \bar{7}\ \overline{10}),$$

$$P^* = (4\ \bar{1})(\bar{4}\ 1)(5\ 6)(\bar{5}\ \bar{6})(12\ 13)\ (\overline{12}\ \overline{13})(9)(\bar{9})(10\ 11)(\overline{10}\ \overline{11})(2\ \bar{3})(\bar{2}\ 3)(7\ 8)(\bar{7}\ \bar{8}),$$

$$S^*P^* = (10\ 8)(\bar{7}\ \overline{11})(1\ 12\ 11\ 13\ \bar{3}\ 3\ 9\ \bar{1}\ \bar{5}\ \bar{8}\ \bar{6})(\overline{13}\ 4\ 5\ 7\ 6\ \bar{4}\ \bar{9}\ 2\ \bar{2}\ \overline{12}\ \overline{10}).$$

The two 2-cycles in S^*P^* describe the orbit whose normal approximation is shown in (a) below. The 11-cycles describe the orbit in (b).



(a) (b)

Note that both orbits are positively turning. The orbit in (a) is simple while the one in (b) is not.

3. Induced assemblages. Let F: $M \to N$ be a normal polymersion with $F(\delta M) = f$. Let $\delta M_1, \ldots, \delta M_\rho$ and $f_1, \ldots, f_\rho$ be the connected components in δM and f, respectively. An *induced assemblage on* f is one determined by F as follows.

Choose orientations on the boundary components, δM_i, $i = 1, \ldots, \rho$, at random. The orientations on the f_i are then determined by F in the natural way. Observe that every curve f_i in f is orientation preserving. It follows from the classification of compact bordered surfaces that M can be obtained from a closed surface by removal of disks. Thus the curves δM_i are preserving. If $F(\delta M_i) = f_i$ is reversing, then a small annulus about δM_i would map to N in such a way that continuity is impossible. Note that this result depends on the assumption that f_i is normal.

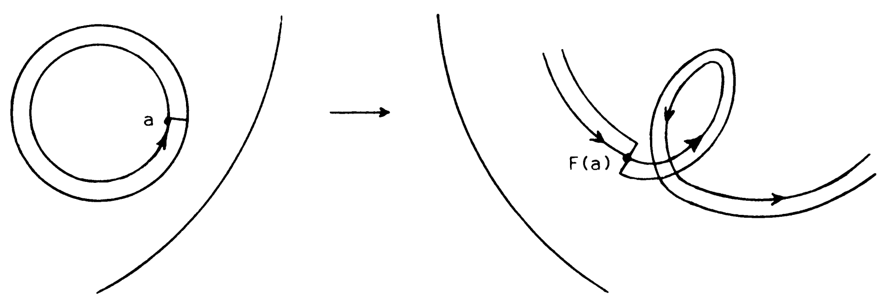


FIGURE 4

w̃ w w

Call the side of f to which F maps points in the inside. Choose any sufficient raying R for f which includes one ray from each branch point. Choose wheels, $w_1, \ldots, w_\beta$, one for each preimage of ∞. Pick a 1-1 correspondence between the preimages of ∞, $\tilde{\infty}_1, \ldots, \tilde{\infty}_\beta$, and the wheels. Let $\tilde{w}_i$, $i = 1, \ldots, \beta$, be the preimage of w_i about $\tilde{\infty}_i$ and let $\tilde{W} = \{\tilde{w}_i, \ldots, \tilde{w}_\beta\}$. Define the successor permutation S as before.

Define the assembling permutation P and a graph G_A in M as follows. Let x on a ray r be a negative crossing with preimage $\tilde{x}$. Lift r from $\tilde{x}$ until it terminates at a point $\tilde{y}$ on $\delta M \cup \tilde{W}$. See Figure 5. The point $\tilde{y}$ is necessarily a preimage of a positive crossing by the choice of the inside of f. Define (xy) to be a cycle of P and the arc $[\tilde{x}, \tilde{y}]$ to be an edge in G_A with the endpoints $\tilde{x}$ and $\tilde{y}$ as vertices. Choose an oriented disk D containing $[\tilde{x}, \tilde{y}]$ and orient its image $F(D)$ so that $F|D$ is sense preserving. Use this orientation on $F(D)$ to determine the auxiliary signs on x and y.

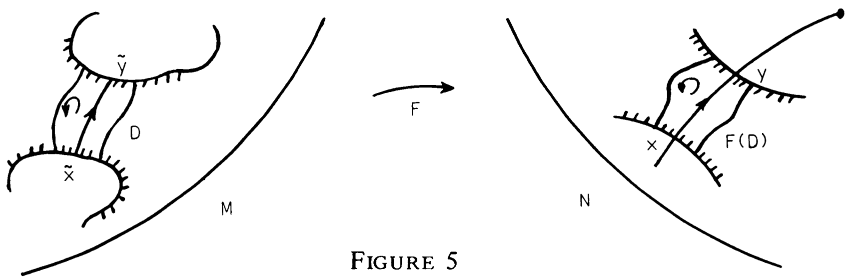


FIGURE 5

Let b be a branch point in N with preimage $\tilde{b}$. Lift the ray initiating at b from $\tilde{b}$ until it terminates at points $\tilde{x}_1, \ldots, \tilde{x}_n$ on $\delta M \cup \tilde{W}$. See Figure 6. These points are necessarily preimages of positive crossings $x_1, \ldots, x_n$ on $f \cup W$. Select an oriented disk D containing $\tilde{b}$ and $\tilde{x}_1, \ldots, \tilde{x}_n$. Although $F|D$ is not now one-one (as was the case with the pairs), $F(D)$ is still a disk. Orient $F(D)$ so that $F|D$ is sense preserving. Use this orientation on $F(D)$ to determine the auxiliary signs on $x_1, \ldots, x_n$. Call the orientation on D counterclockwise and let $\tilde{x}_1, \ldots, \tilde{x}_n$ be the clockwise ordering of these points about $\tilde{b}$. Then $(x_1 x_2 \ldots x_n)$ is defined to be a cycle of P. Define the arcs $[\tilde{b}, \tilde{x}_i]$ and the points $\tilde{b}$, $\tilde{x}_i$, $i = 1, \ldots, n$, to be edges and vertices of G_A. Include any remaining positive crossings as singleton cycles in P and their preimages as vertices in G. Finally, all arcs on $\delta M \cup \tilde{W}$ connecting preimages of crossings are included as edges in G_A.

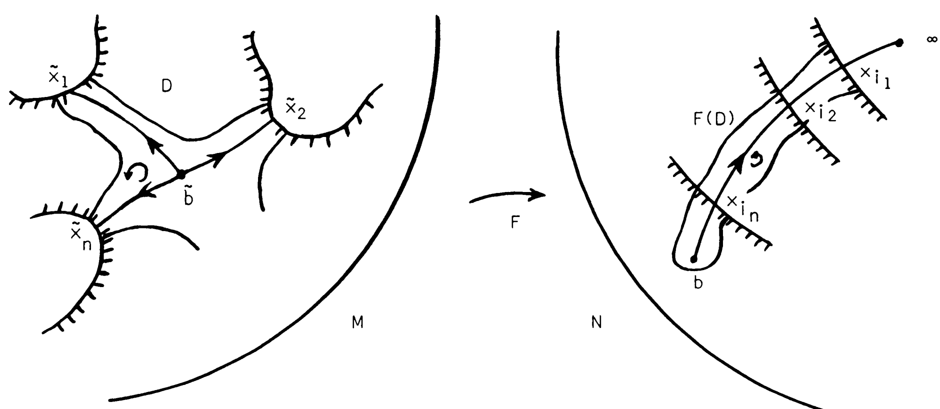


FIGURE 6

Note we have observed that the lift of a ray from a negative crossing terminates at a positive. Likewise, the lift from a positive crossing on a fundamental ray terminates at a negative. Hence we have a 1-1 correspondence between the positive and negative crossings on fundamental rays and have the following observation.

(2) If $f = F(\delta M)$ is the boundary image of a normal polymersion, then f satisfies the generator parity condition.

Define the assemblage induced by F, A_F, to be (R, W, S, P) as chosen above. From these choices it follows that:

(3) The images of the face boundaries in $G_A(M)$ are the orbits of A_F.

3.1. THEOREM. *Induced assemblages are transitive, effective, and simple.*

PROOF. The fact that induced assemblages are transitive and effective follows directly from the choice of the assemblage. The transitivity follows from the connectedness of the surface M and the effectiveness from the choice of P.

To see that the orbits are simple, first observe that the orbits are positively turning and lie in $N - \alpha$. (This was observation (1).) Let M_1 be a face in the imbedding of $G_A(M)$ and $F(\delta M_1) = 0$, the corresponding orbit. Let F_1 be F restricted to M_1. Since F_1 is a polymersion mapping into a disk, it follows that M_1 is orientable. If not, we could construct (with sufficient smoothing) a normal polymersion F_1': $M_1' \to N - \alpha$, where M_1' is nonorientable. Compose with a homeomorphism h: $N - \alpha \to D$, where D is a disk in the sphere S^2. We now have a map F_1'': $M_1' \to S^2$ where $F_1''(\delta M_1')$ is a normal set of curves in S^2. Hence there is an extension F_2: $M_2 \to S^2$ where M_2 is orientable. See [3] or [4]. Identify M_1' and M_2 along their boundaries to obtain a nonorientable surface $\overline{M}$ and a map $\overline{F}$: $\overline{M} \to S^2$, where $\overline{F}$ restricted to M_1' and M_2 is F_1'' and F_2, respectively. It follows from the definition of polymersion and the compactness of the surfaces that $(\overline{F}, \overline{M})$ is a branched cover of S^2. We have a contradiction since $\overline{M}$ is nonorientable.

We have a normal polymersion F_1': $M_1' \to N - \alpha$ where the images of the boundary curves are positively turning relative to the choice of the inside. Now choose orientations on M_1' and $N - \alpha$ so that F_1' is sense preserving. Reorient the boundary curves on M_1' if necessary to agree with the orientation on M'. Reorient the curves in $N - \alpha$ if necessary to agree with F_1'. The fact that the curves in $F_1'(M_1')$ were positively turning (as defined here) implies that they are positively turning curves in the disk $N - \alpha$ in the usual sense. The result is that we have a sense preserving, normal polymersion with no branch points from a orientable surface to the disk where the images of the boundary components are positively turning curves. The only possibility is that M_1' is a disk and the image of its boundary is a simple closed curve. (This last fact is shown in observation (5) of [3].)

4. The tangent winding number. The results in the next section make use of the tangent winding number of a curve in a surface defined relative to a vector field on that surface. Let f be a two-sided regular closed curve in a surface N with a vector field whose zeroes (if any) do not occur on f. Choose a side of f to be the inside. Since f is two-sided, a neighborhood V of f will be orientable. Choose a direction on f and orient V so that the inside of f lies to the left. The *tangent winding number* *of f*, twn(f), is the number of times a vector tangent to f rotates relative to the vector field as f is traversed one time. A counterclockwise rotation is considered positive. See [**1**]. If $f = \{f_1, \ldots, f_\rho\}$, then twn($f$) is defined to be $\Sigma\{\text{twn}(f_i) | i = 1, \ldots, \rho\}$. Note the following.

(4) The choice of the inside and not the direction of the curve determines the twn. If the direction of the curve is reversed, the orientation on the neighborhood is also reversed to keep the inside to the left. Hence the twn is unchanged.

Note that if a curve f in N satisfies GPC, then twn(f) can be calculated by decomposing f into simple closed curves lying in $N - \alpha$. First pair each negative crossing on a ray in α with a positive and modify to obtain a curve f' as shown in Figure 7. The modification in (a) lowers the twn by one and that in (b) raises it by one (provided there are no zeros in the neighborhood U of the ray segment). Hence if f' results from n modifications of type (a) and m of type (b) then $\text{twn}(f) = \text{twn}(f') + n - m$.

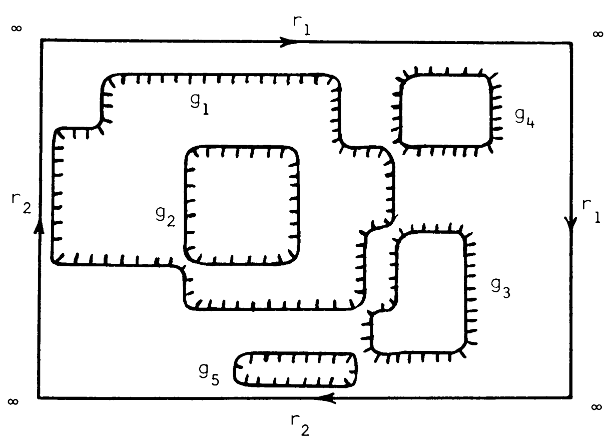
Now decompose f' into Gaussian circles $g_1, \ldots, g_n$. It follows that $\text{twn}(f') = \Sigma\{\text{twn}(g_i) | i = 1, \ldots, n\}$. The tangent winding number of a simple closed curve g bounding a disk D is

(i) $1 - S$, if D lies to the inside and

(ii) $S - 1$, if D lies to the outside,

where S is the sum of the indices of the zeroes in D. Hence twn(g_i), $i = 1, \ldots, n$, can be easily calculated.

4.1. Example. The modifications above applied to the curve in 2.1 yield the simple closed curves below.



Relative to a vector field with no zeroes we have

$$\begin{aligned}\text{twn}(f) &= \text{twn}(f') - 1\\ &= \text{twn}(g_1) + \text{twn}(g_2) + \text{twn}(g_3) + \text{twn}(g_4) + \text{twn}(g_5) - 1\\ &= 1 + 1 - 1 - 1 + 1 - 1 = 0.\end{aligned}$$

Relative to a vector field with 2 zeroes, one of index 1 bounded by both g_1 and g_2 and one of index -1 bounded by g_3 we have

$$\begin{aligned}\operatorname{twn}(f) &= \operatorname{twn}(f') - 1 \\ &= \operatorname{twn}(g_1) + \operatorname{twn}(g_2) + \operatorname{twn}(g_3) + \operatorname{twn}(g_4) + \operatorname{twn}(g_5) - 1 \\ &= 0 + 0 - 2 - 1 + 1 - 1 = -3.\end{aligned}$$

Although an orbit of an assemblage is not a regular curve, we define the tangent winding number of an orbit to be the limit of the tangent winding numbers of an approaching sequence of regular curves. It follows from this definition and (4) above that if A_f is a transitive effective assemblage for f with wheel set W and orbits $O_1, \ldots, O_\zeta$, then

(5) $\operatorname{twn}(f \cup W) = \Sigma\{\operatorname{twn}(O_i) | i = 1, \ldots, \zeta\} - \nu$.

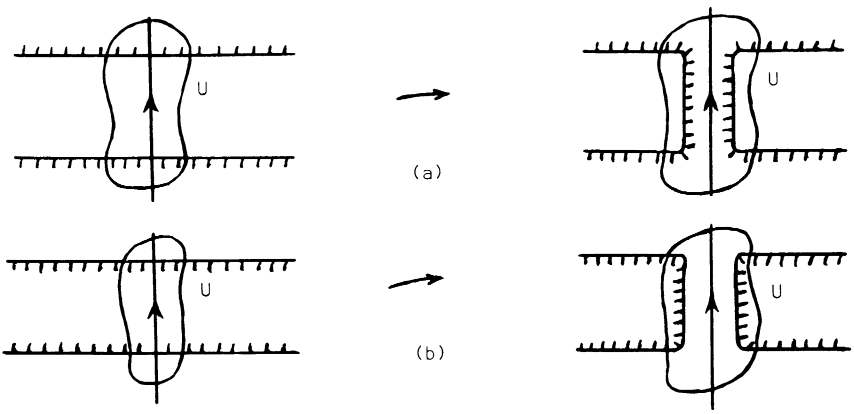


FIGURE 7

Furthermore, if S is the sum of the indices of the zeroes in the disk bounded by the innermost wheel w, then $\operatorname{twn}(w) = S - 1$ and $\operatorname{twn}(W) = \beta(S - 1)$, where β is the number of wheels. (Recall that wheels are concentric and the disk bounded by a wheel lies to the outside.) If all the zeroes of the vector field on N are bounded by each wheel, then $S = \chi(N)$ (the Euler characteristic of N) by the Poincare-Hopf index theorem (see [5]) and we have

(6) $\operatorname{twn}(W) = \beta(\chi(N) - 1)$.

Facts (5) and (6) were used in the proof of 2.4 in [3] and also hold in the case where N is nonorientable.

5. Existence and representation and polymersions. The following theorem shows that the conditions in 3.1 are sufficient for the existence of extensions of normal curves and also establishes a combinatorial representation for polymersions between arbitrary closed surfaces. We use the letter μ for the total multiplicity of a polymersion; i.e. the sum of the multiplicities of all the critical points. The tangent winding numbers will be calculated relative to a vector field vanishing only at a finite number of points in a neighborhood $U \subset N - (f \cup W)$ of ∞.

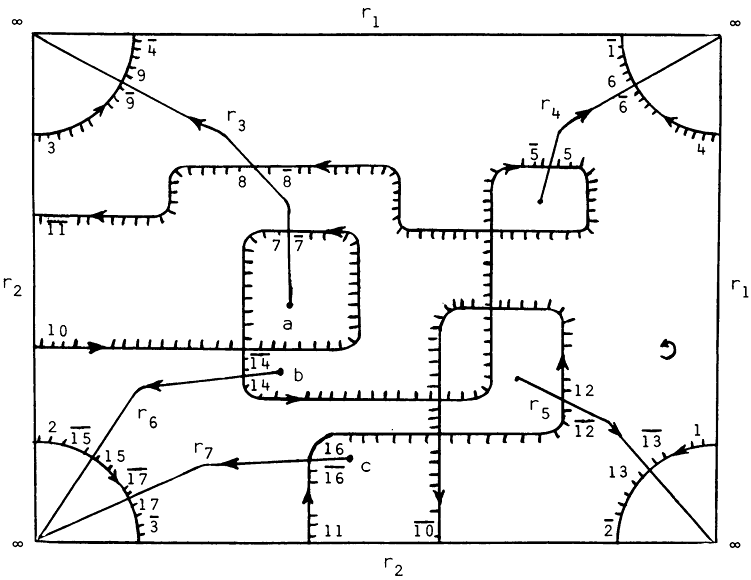
5.1. THEOREM. *Let f be a set of curves in a surface N satisfying GPC and let $A = (R, W, S, P)$ be a transitive, effective assemblage with β wheels, ν negative crossings, ζ orbits, and with partial branching number π. Then f has a polymersion extension F: $M \to N$ from a connected surface M of characteristic $\chi(M)$ where*

(i) $\mu(F) = \nu + \operatorname{twn}(f) - \zeta + \pi + \beta(\chi(N) - 1)$,
(ii) $\chi(M) = \zeta - \nu - \pi + \beta$.

If in addition A is simple, then A identifies (up to topological equivalence) a particular map F_A: $M_A \to N$ with $\mu(F_A) = \pi$ and $\chi(M_A) = \zeta - \nu - \pi + \beta$. If A is an assemblage induced by some polymersion F, then F_A is equivalent to F.

The proof is almost line for line the same as the proof of 2.4 in [3] and will not be included here in detail. In the case where A is simple, the surface M_A is constructed from cells corresponding to the simple orbits in A and F_A defined from homeomorphisms on cells. The multiplicity is μ from the construction and $\chi(M_A)$ is determined from the Euler formula by counting vertices, edges, and faces. In the case where A is induced, then M_A is a duplicate of M since the orbits are images of face boundaries of $G_A(M)$. In the case where the assemblage is not simple, the orbits are still positively turning and the proof in [3] applies.

5.2. Example. Let f be the curve of Example 2.1. Choose an orientation on $N - \alpha$ to determine the auxiliary signs on the interior crossings. Add rays r_6 and r_7 and extend the original permutations S and P as shown below.



The cycles in the permutations S, P, S^*, P^*, and S^*P^* are

$$S = (10\ 7\ 14\ 5\ 8\ 11\ 16\ 12)(1\ 13\ 2\ 15\ 17\ 3\ 9\ 4\ 6),$$

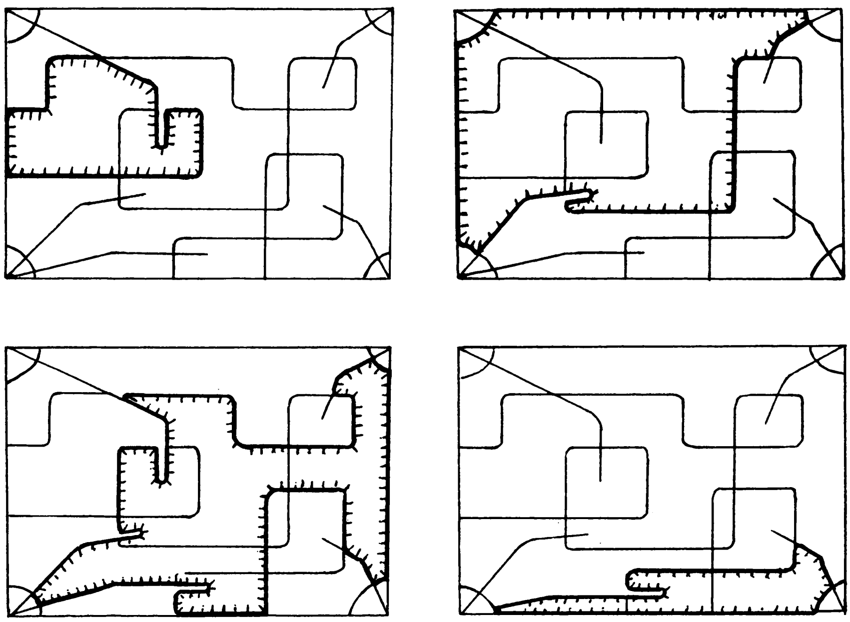
$$P = (4\ 1)(5\ 6)(12\ 13)(9)(10\ 11)(2\ 3)(7\ 8)(14\ 15)(16\ 17),$$

$$S^* = \left[\begin{array}{l}(10\ 7\ 14\ 5\ 8\ 11\ 16\ 12)(1\ 13\ 2\ 15\ 17\ 3\ 9\ 4\ 6)\\ (\overline{12}\ \overline{16}\ \overline{11}\ \bar{8}\ \bar{5}\ \overline{14}\ \bar{7}\ \overline{10})(\bar{6}\ \bar{4}\ \bar{9}\ \bar{3}\ \overline{17}\ \overline{15}\ \bar{2}\ \overline{13}\ \bar{1}),\end{array}\right.$$

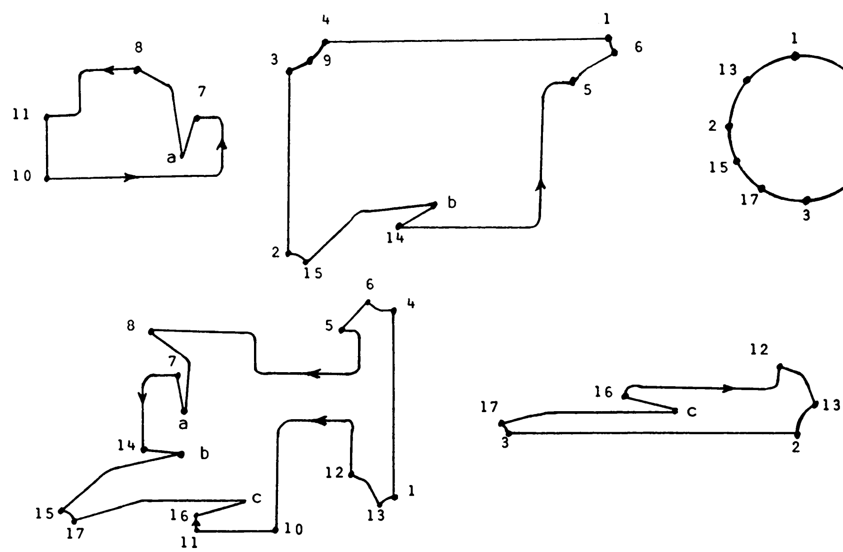
$$P^* = \left[\begin{array}{l}(4\ \bar{1})(5\ 6)(12\ 13)(9)(10\ 11)(2\ \bar{3})(7\ 8)(14\ 15)(16\ \overline{17})\\ (\bar{4}\ 1)(\bar{5}\ \bar{6})(\overline{12}\ \overline{13})(\bar{9})(\overline{10}\ \overline{11})(\bar{2}\ 3)(\bar{8}\ \bar{7})(\overline{15}\ \overline{14})(\overline{16}\ 17),\end{array}\right.$$

$$S^*P^* = \left[\begin{array}{l}(10\ 8)(1\ 12\ 11\ \overline{17}\ \overline{14}\ \bar{8}\ \bar{6})(\bar{1}\ \bar{5}\ \overline{15}\ 3\ 9)(13\ \bar{3}\ 16)\\ (\bar{7}\ \overline{11})(\overline{13}\ 4\ 5\ 7\ 15\ \overline{16}\ \overline{10})(6\ \bar{4}\ \bar{9}\ 2\ 14)(\bar{2}\ \overline{12}\ 17).\end{array}\right.$$

There are four orbits corresponding to the cycles in S^*P^*.



Note that all the orbits above are simple. The cells used to construct M_A are below.



The surface M_A is constructed by identifying like edges. Since A is simple, $\mu(F_A) = \pi = 3$. $\chi(M_A) = \zeta - \nu - \pi + \beta = 4 - 5 - 3 + 1 = -3$.

6. Extensions of normal curves in compact surfaces. Some results on extensions of normal curves will be presented using Theorem 5.1.

6.1. THEOREM. *A normal set of two-sided regular curves f in a closed surface has a polymersion extension if and only if there exists a choice of sides for the components of f such that relative to these choices, f satisfies GPC.*

PROOF. The necessity of this condition has already been observed. Suppose we have a set of curves f satisfying the condition above. We wll show that there exists a transitive, effective, assemblage on f. Choose a sufficient raying with generator rays α. Pair each negative crossing with a positive crossing further out on the same ray where possible. This will (possibly) leave unpaired negative crossings in $N - \alpha$ and positive-negative pairs on rays in α where the positive precedes the negative. Call these pairs *negative pairs*. Add one wheel for each unpaired negative and one for each negative pair. Define P on these crossings by pairing the negatives with positives from the wheels and the positives with negatives from the wheels. This exhausts the negative crossings. Define P arbitrarily on the remaining positive crossings. The result is an effective assemblage. If this assemblage is not transitive, add rays if necessary and pair crossings on each component curve with crossings on a single wheel.

6.2. THEOREM. *Every normal polymersion F: $M \to N$ can be extended to a branched cover $\bar{F}$: $\bar{M} \to N$ from some closed surface $\bar{M}$.*

PROOF. By 6.1, $F(\delta M)$ is a normal set of curves satisfying GPC. Also by 6.1 there is an extension F': $M' \to N$. Identify M and M' along their boundaries to obtain $\bar{M}$. Define $\bar{F}$ to be f restricted to M and F' restricted to M'. It follows from the compactness of the surfaces and the definition of polymersion that $\bar{F}$ is a branched cover.

6.3. THEOREM. *If F: $M \to N$ is a polymersion where N is orientable, then M is orientable.*

PROOF. By 6.2, F can be extended to a branched cover $\bar{F}$: $\bar{M} \to N$. If M were nonorientable, then $\bar{M}$ would be. But there are no branched covers with nonorientable domains and orientable targets.

6.4. THEOREM. *Let F: $M \to N$ be a polymersion from a bordered surface M to a smooth, closed surface N. Let $F(\delta M) = f$. Then relative to a vector field vanishing only at a finite number of points in a neighborhood $U \subset N - f$ of some fixed point ∞,*

$$\chi(M) + \mu(F) = \operatorname{twn}(f) + \beta \cdot \chi(N),$$

where $\beta = |F^{-1}(\infty)|$.

PROOF. Suppose the assemblage induced by F has ν negative crossings, ζ orbits, β wheels, and partial branching number π. Then from 5.1,

(i) $\mu(F) = \nu + \operatorname{twn}(f) - \zeta + \pi + \beta(\chi(N) - 1)$, and

(ii) $\chi(M) = \zeta - \nu - \pi + \beta$.

Add lines (i) and (ii) above to obtain the result.

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